small, we can treat the body as a continuous distribution of mass. We subdivide the body into n small elements of mass; $\Delta m_1, \Delta m_2 \ldots \Delta m_n$; the i^{th} element Δm_i is taken to be located about the point (x_i, y_i, z_i). The coordinates of the centre of mass are then approximately given by

$$X = \frac{\sum(\Delta m_i)x_i}{\sum \Delta m_i}, Y = \frac{\sum(\Delta m_i)y_i}{\sum \Delta m_i}, Z = \frac{\sum(\Delta m_i)z_i}{\sum \Delta m_i}$$

As we make n bigger and bigger and each Δm_i smaller and smaller, these expressions become exact. In that case, we denote the sums over i by integrals. Thus,

$$\sum \Delta m_i \to \int dm = M,$$

$$\sum (\Delta m_i)x_i \to \int x\,dm,$$

$$\sum (\Delta m_i)y_i \to \int y\,dm,$$

and $$\sum (\Delta m_i)z_i \to \int z\,dm$$

Here M is the total mass of the body. The coordinates of the centre of mass now are

There formulae are used for continious rigid bodies.

$$X = \frac{1}{M}\int x\,dm, Y = \frac{1}{M}\int y\,dm \text{ and } Z = \frac{1}{M}\int z\,dm \quad (7.5a)$$

The vector expression equivalent to these three scalar expressions is

$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\,dm \quad (7.5b)$$

If we choose, the centre of mass as the origin of our coordinate system,

$$\mathbf{R} = \mathbf{0}$$

i.e., $\int \mathbf{r}\,dm = \mathbf{0}$

$$\text{or } \int x\,dm = \int y\,dm = \int z\,dm = 0 \quad (7.6)$$

Often we have to calculate the centre of mass of homogeneous bodies of regular shapes like rings, discs, spheres, rods etc. (By a homogeneous body we mean a body with uniformly distributed mass.) By using symmetry consideration, we can easily show that the centres of mass of these bodies lie at their geometric centres.

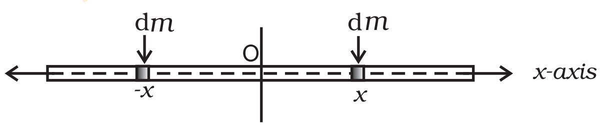

Fig. 7.8 *Determining the CM of a thin rod.*

Let us consider a thin rod, whose width and breath (in case the cross section of the rod is rectangular) or radius (in case the cross section of the rod is cylindrical) is much smaller than its length. Taking the origin to be at the geometric centre of the rod and x-axis to be along the length of the rod, we can say that on account of reflection symmetry, for every element dm of the rod at x, there is an element of the same mass dm located at $-x$ (Fig. 7.8).

The net contribution of every such pair to the integral and hence the integral $\int x\,dm$ itself is zero. From Eq. (7.6), the point for which the integral itself is zero, is the centre of mass. Thus, the centre of mass of a homogenous thin rod coincides with its geometric centre. This can be understood on the basis of reflection symmetry.

{ Very Imp.

This result can be generalised for every homogeneous continious rigid bodies.

The same symmetry argument will apply to homogeneous rings, discs, spheres, or even thick rods of circular or rectangular cross section. For all such bodies you will realise that for every element dm at a point (x, y, z) one can always take an element of the same mass at the point $(-x, -y, -z)$. (In other words, the origin is a point of reflection symmetry for these bodies.) As a result, the integrals in Eq. (7.5 a) all are zero. This means that for all the above bodies, their centre of mass coincides with their geometric centre.

Example 7.1 Find the centre of mass of three particles at the vertices of an equilateral triangle. The masses of the particles are 100g, 150g, and 200g respectively. Each side of the equilateral triangle is 0.5m long.

Answer

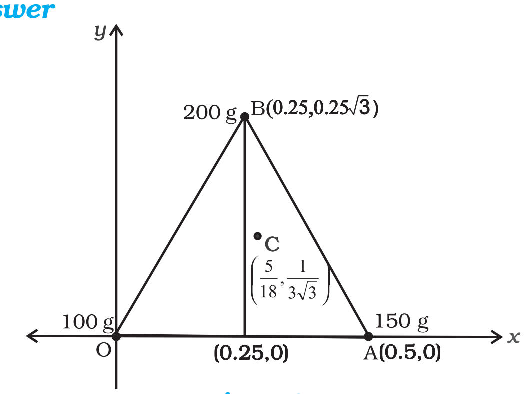

Fig. 7.9